The instability to long waves of unbounded parallel inviscid flow

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Formulas for the determination of the instability characteristics of unbounded parallel flow are obtained for the case of long waves, and applied, together with some general results, to give a qualitative description of the different modes of instability of such flows. It is found that there is a finite number of different modes unstable to long waves, essentially one for each relative maximum and minimum of the velocity profile. These modes appear to become stable when the wavelength is sufficiently small, reducing to neutral solutions associated with inflexion points as stability is approached. The formulas are also useful for quantitative calculation of instability characteristics.

1. Introduction

The stability of a parallel flow of a non-viscous incompressible fluid has been studied since last century, principally by examining the growth properties of an infinitesimal wave-like perturbation of a basic flow with given velocity

$$\mathbf{u}_* = \{w_*(y_*), 0, 0\}.$$

By a simple transformation (Squire 1933) the problem for three-dimensional waves can be reduced to an equivalent two-dimensional problem, and in fact the least stable disturbances are already two-dimensional. It is thus sufficient for many purposes to consider the two-dimensional case. The stability problem, of course, really involves the question of the growth properties of an *arbitrary* infinitesimal perturbation, but it has generally been assumed that such perturbations can be resolved into independent components of wave-like character, i.e. having an exponential time factor (possibly with complex exponent) $e^{-i\alpha ct}$ and a pure imaginary $e^{i\alpha x}$ x-factor, multiplied by a function of y. Each such component is supposed to satisfy the linearized equations of motion and boundary conditions.

Some care must be used in this approach because of the occurrence of 'improper' modes associated with concentrated layers of vorticity, and the corresponding continuous part of the c-spectrum, as well as the ordinary stable or unstable waves of the discrete c-spectrum. The existence of the continuous spectrum was known already to Rayleigh, but its importance, particularly in connexion with the initial-value problem, has been emphasized recently by Case, in some interesting papers (Case 1960*a*, *b*, *c*; 1961). In this connexion one should also mention (i) the work of Orr (1907, pp. 26, 27) and Friedrichs (1942, p. 209) on a class of solutions for the plane Couette-flow stability problem which clearly shows the inadequacy of considering only the discrete modes, (ii) the perceptive discussion of Orr's work by Synge (1933, p. 15), (iii) the detailed study by Eliassen, Höiland & Riis (1953) of the initial value problem for stability of plane Couette flow in a stratified fluid, and (iv) the solution by Carrier & Chang (1959) of the initial value problem in Rayleigh–Taylor instability, where a similar situation occurs.

On the other hand, important as the continuous spectrum is for an understanding of the initial value problem, it still appears that those modes leading to actual instability are associated with the discrete spectrum, and it is in these that we are particularly interested here. Our main purposes are to give some formulas and techniques by which an interesting portion of the instability characteristics of an unbounded parallel flow can be readily determined, in the case of a more or less arbitrary velocity profile, and to discuss qualitatively the nature of the long wave instabilities of such flows. We shall be concerned only with discrete modes, and formulate the problem as Lord Rayleigh originally did in 1887 (cf. Rayleigh 1945).

Supposing the basic flow $w_*(y_*)$ to be dimensionally characterised by some length scale L and velocity scale V; choose dimensionless variables of time, position, and velocity in the usual way

$$t = t_* V/L, \quad \mathbf{x} = \mathbf{x}_*/L, \quad \mathbf{u} = \mathbf{u}_*/V$$

and $w = w_*/V$. Each wave component then has a stream function

$$\psi_* = \phi_*(y_*) \exp\{i\alpha_*(x_* - c_* t_*)\}$$

or, in dimensionless form,

$$\psi = \phi(y) \exp\{i\alpha(x - ct)\},$$

where $c = c_r + ic_i$ is a complex wave velocity and α a real positive wave-number. Thus c_r is the phase velocity and $\exp(\alpha c_i t)$ the growth factor of the wave, and the basic flow is stable, neutrally stable, or unstable to this wave according as c_i is negative, zero, or positive, respectively.

The linearized equations of motion now lead to the Rayleigh stability equation,

$$(w - c)(\phi'' - \alpha^2\phi) - w''\phi = 0, \tag{1.1}$$

where primes denote differentiations with respect to y. If the flow is bounded by parallel rigid walls at $y = y_1$ and y_2 the boundary conditions of vanishing normal velocity give

$$\alpha\phi = 0 \quad (y = y_1, y_2). \tag{1.2}$$

In the case of unbounded or semi-bounded flow the same boundary conditions are appropriate, with y_1 and/or y_2 infinite. In the case of a free surface of constant pressure it can be shown that the appropriate boundary condition is

$$\phi' = 0. \tag{1.3}$$

This completes the statement of the eigenvalue problem whose solution determines the function $c_i = c_i(\alpha)$ and hence the stability characteristics of the flow. Most of the studies which have been made of this problem have been for special velocity profiles $w(y)$; however, Tollmien (1935) has shown that in many ordinary circumstances if $w(y)$ has an inflexion point at $y = y_s$ in the flow domain, then a neutrally stable eigensolution $\phi = \phi_s$, $\alpha = \alpha_s \neq 0$, $c = w(y_s) \equiv w_s$, exists. Various authors have computed solutions to this Sturm–Liouville problem by variational methods. Tollmien and later Lin (cf. Lin 1955) showed that unstable solutions existed near this one, and Lin obtained the formula

$$\left(\frac{\partial c}{\partial \alpha^2}\right)_{\alpha=\alpha_s} = -\left\{\int_{y_1}^{y_2} \phi_s^2 dy\right\} \Big/ \left\{\mathfrak{P} \int_{y_1}^{y_2} \frac{w'' \phi_s^2}{(w-w_s)^2} dy + i\pi \left(\frac{w''' \phi_s^2}{w'^2}\right)_{y=y_s}\right\}, \tag{1.4}$$

where $\mathfrak{P}$ denotes the principal value of the integral. This gives the form of the curve $c = c(\alpha)$ in the complex c plane near $c = w_s$, once the neutral eigensolution is known. In this paper we obtain (in the case of unbounded flow) expansions for small α of the eigenfunction and eigenvalue, for an essentially arbitrary velocity profile. In many cases the radius of convergence of these expansions seems to be large enough that by combining equation (1.4) for α near α_s with our results for α near 0 a considerable portion of the instability characteristics of the flow can be analytically determined.

Our principal interest in this paper is in the case of unbounded flows, and to some extent semi-bounded flows. Previous work has been mainly on various specific representations of the velocity profiles of jets and half-jets. Carrier (cf. Esch 1957) has compared the stability characteristics of the half-jet profiles

$$w = y/|y|, \tag{1.5a}$$

$$\left.\begin{aligned} w &= y/|y| \quad (|y| > 1) \\ &= y \qquad\; (|y| < 1), \end{aligned}\right\} \tag{1.5b}$$

and

$$w = \operatorname{erf} y. \tag{1.5c}$$

As $\alpha \to 0$, $c_i \to \pm 1$ for these profiles, and the last two have remarkably similar functions $c_i(\alpha)$. Instability characteristics of simple representations of the full jet profile also resemble one another.

The physical explanation of these similarities is that the detailed structure of the velocity profile is not important for small α. On the large scale of a long wave, the velocity profile looks essentially like a sharp discontinuity between its values at $\pm\infty$. To state this a little more precisely, we take a fixed $w(y)$ and look at it in the dimensional form

$$w_*(y_*) = Vw(y_*/L).$$

We may choose the origin of velocity (by making a Galilei transformation if necessary) so that

$$w(\infty) + w(-\infty) = 0.$$

This may change c_r, but not c_i. Furthermore, if $w_*(\infty) \neq 0$, we choose $V = w_*(\infty)$, i.e. $w(\infty) = 1$. We now hold α_* fixed and let $L \to 0$. We then get a velocity profile which in physical variables looks like the Helmholtz flow (1.5a), for which it is known that $c = \pm i$. We deduce that $c \to \pm i$ as $\alpha \to 0$ for any $w(y)$ with

$w(\infty) = -w(-\infty) = 1$, because c is a function of $\alpha = \alpha_* L$. (This assumes that the stability theory is physically reasonable so that this limit makes sense.) If $w_*(\infty) = 0$ we are obliged to choose V differently, and in this case the limiting velocity profile is zero, a neutrally stable flow with $c = 0$. In this case we thus expect that $c \to 0$ as $\alpha \to 0$, although the argument here seems a bit weaker because of the somewhat more obvious non-uniformity in the limiting process. At any rate, these considerations give a physical rationalization to the results of Carrier's comparison, for small α. The more detailed study of the instability characteristics of unbounded flow for small α is taken up in the remainder of this paper. We note here, however, that in addition to the types of instability just described, in general there are also other modes, for which c behaves differently as $\alpha \to 0$ (cf. §6).

When $\alpha = 0$ there is a 'trivial' eigensolution, $\phi = A(w-c)$, of the Rayleigh equation. It is fairly well known (but apparently unpublished) that this solution is really a form of the basic flow. For the total x-component of the velocity of the perturbed flow is

$$u = w(y) + \partial\psi/\partial y = w(y) + Aw'(y),$$
$$= w(y+A) + O(A^2),$$

and the y-component $v = -\partial\psi/\partial x = 0$. Thus the trivial solution is really the basic parallel flow displaced laterally by the small distance A. In fact, for any solution it is readily seen that the vertical displacement at (x, t) of the material surface with mean level y is

$$\eta = F(y) \exp\{i\alpha(x-ct)\}$$

where the amplitude $F \equiv \phi/(w-c)$. In terms of F the Rayleigh stability equation (1.1) becomes

$$[(w-c)^2 F']' = \alpha^2 (w-c)^2 F. \tag{1.6}$$

The trivial solution is just $F = \text{constant}$, $\alpha = 0$.

Using a power series in α^2, the first term of which was a linear combination of the trivial solution and the other solution $(w-c)\int(w-c)^{-2}dy$ of the Rayleigh equation for $\alpha = 0$, Heisenberg (cf. Lin 1955) solved the Rayleigh equation for bounded flows. This method cannot be used for unbounded flows, for the series is not uniformly convergent at $y = \pm\infty$ in this case. (However, Heisenberg did obtain approximate results for semi-bounded flows by using a large finite region with a modified boundary condition.) The reason is that $\phi \sim e^{\mp\alpha y}$ as $y \to \pm\infty$ in order to satisfy the Rayleigh equation and the boundary conditions for all non-zero α, however small. It is possible, however, to avoid this non-uniformity by 'dividing out' the asymptotic behaviour of ϕ for large y, and then working with a power series in α as has been done by Lighthill (1957) and Miles (1957). This approach is similar to that used by Tatsumi & Kakutani (1958) and Tatsumi & Gotoh (1960), in the case of the Orr–Sommerfeld equation.

Another approach, using integral equations, somewhat related to that used by Howard (1959) for the viscous jet problem, is also available. This method, which leads directly to the eigenvalue relation without explicit consideration of the eigenfunctions, has some advantages and will be considered briefly in § 4.

We emphasize that in this paper we are entirely concerned with the *inviscid* theory of hydrodynamic stability, in which stability is indicated by *real* c, instability by *complex* c; the Rayleigh equation being invariant under complex conjugation except for a reversal of the sign of c_i, any solution of it with $c_i < 0$ implies the existence of another solution (the complex conjugate) with $c_i > 0$, and hence instability. The only kind of stability possible on the inviscid theory is *neutral* stability, with $c_i = 0$. Since the Orr–Sommerfeld equation is not invariant in this way under complex conjugation, similar remarks do not apply in the viscous case, and the relation of the solutions of the Rayleigh equation to the limits as $\nu \to 0$ of solutions of the Orr–Sommerfeld equation requires special consideration. For this we may refer to Lin (1955, Ch. 8), and references given there, where it is shown that at least for analytic $w(y)$ a solution of the Rayleigh equation with $c_i > 0$ *is* a limit of a solution of the Orr–Sommerfeld equation, though its complex conjugate may not be, throughout the entire flow domain. In this paper we shall refer to 'instability' as '$c_i > 0$', but it should be remembered that on the inviscid theory $c_i < 0$ equally implies instability. The rather subtle mathematical questions raised by the inclusion of viscosity, important as they are in some cases, are not considered here, and our results must be taken with this in mind. Nevertheless, a full understanding of the inviscid theory is a desirable preliminary to any study of the viscous case.

We conclude this introductory section with an outline of the rest of the paper:

§2. Here, proceeding formally by the direct approach through the differential equation, we derive several formulas for the eigenfunctions and eigenvalue relation, in terms of the expansions in powers of α mentioned above.

§3 gives a proof of convergence of the series for the eigenfunction.

§4 presents the integral equation approach, which leads directly to a slightly different form of the eigenvalue relation. We also discuss briefly some convergence questions not considered in §3.

§5 is a list of examples of known exact solutions. We give these partly to provide illustrations in which the results of our small α formulas can be checked, but especially because they provide some motivation for the discussion in §6 of the general nature of the various unstable eigenmodes. We also feel that such a collection of exact solutions will be found generally convenient.

§6 contains our main theoretical results. (Readers not interested in the details of the derivations of the formulas may omit §§2–4, and skip at once to §5, or even §6.) We begin with some general results not restricted to long waves, and then show that for unbounded flow only certain types of instability can occur in the limit of small α. Finally, we apply our small-α formulas to show that these possible instabilities *do* occur, and describe them in more detail.

§7 illustrates with two examples the use of the small-α formulas to give quantitative numerical results.

2. The eigenvalues and eigenfunctions for small wave-number

It is convenient to define $W \equiv w - c$, so that the Rayleigh equation becomes

$$W(\phi'' - \alpha^2\phi) - W''\phi = 0. \tag{2.1}$$

We shall suppose that w approaches constant values as $y \to \pm\infty$. If the values are the same, we normalize so that they are zero, and refer to this case as the 'jet case'. If they are different we normalize so that they are ± 1, and call this the 'shear-layer case'.

In both cases, we assume that $w'' \to 0$ at $\pm\infty$ sufficiently rapidly that (at least for $c \neq 0$) equation (2.1) has two solutions asymptotic respectively to $e^{\pm\alpha y}$ as $y \to +\infty$ and also two with these asymptotic properties as $y \to -\infty$. A sufficient condition for this is the convergence of $\int_{-\infty}^{\infty} |W''/W|\, dy$ (Coddington & Levinson 1955, Ch. 3, Theorem 8·1); later we shall even explicitly assume that $w' \to 0$ exponentially at $\pm\infty$. Our appoach is now as follows. For fixed $c(\neq 0)$, we seek the solution of equation (2.1) with asymptotic behaviour $e^{-\alpha y}$ as $y \to +\infty$ in the form

$$\phi_1 = e^{-\alpha y}\chi(y) \quad (y > 0), \tag{2.2a}$$

using for this purpose an expansion of χ into a power series in α,

$$\chi = \sum_0^\infty \alpha^n \chi_n.$$

Similarly, we obtain the solution of equation (2.1) with asymptotic behaviour $e^{\alpha y}$ as $y \to -\infty$ in the form

$$\phi_2 = e^{\alpha y}\theta(y), \tag{2.2b}$$

with $\theta = \sum_0^\infty (-\alpha)^n \theta_n$. We then obtain the eigenvalue relation $F(\alpha, c) = 0$ by requiring that ϕ_1 and ϕ_2 (properly normalized) should in fact be the same solution on $(-\infty, \infty)$. This amounts to requiring that $\phi_1(0+) = K\phi_2(0-)$ and

$$\phi_1'(0+) = K\phi_2'(0-) \quad (K \text{ constant}),$$

since a solution of equation (2.1) is determined by such initial conditions. Actually, by taking successively more terms in the expansions for χ and θ and using these conditions at 0 we shall obtain a sequence of approximations to the eigenvalue relation.

For the determination of χ and θ, there remains of course an arbitrary constant factor. The most convenient normalization is

$$\left.\begin{aligned} \chi(\infty) &= W(\infty) \equiv W_\infty, \\ \theta(-\infty) &= W(-\infty) \equiv W_{-\infty}, \text{ i.e.} \\ \chi_0(\infty) &= W_\infty, \quad \chi_n(\infty) = 0, \\ \theta_0(-\infty) &= W_{-\infty}, \quad \theta_n(-\infty) = 0 \quad (n \geqslant 1). \end{aligned}\right\} \tag{2.3}$$

We shall for the most part not write down explicitly the formulas for θ; they are essentially the same as for χ except for certain obvious changes of sign which will be pointed out.

Inserting (2.2*a*) into (2.1) we obtain for χ the equation

$$W\chi'' - W''\chi = 2\alpha W\chi' \quad (y > 0). \tag{2.4}$$

The same equation with α replaced by $-\alpha$ holds for θ. On inserting the power series for χ this gives

$$\left.\begin{aligned} \{W^2(\chi_0/W)'\}' &= 0, \\ \{W^2(\chi_{n+1}/W)'\}' &= 2W\chi_n' \quad (n \geqslant 0). \end{aligned}\right\} \tag{2.5}$$

The equations for the θ_n are the same as these. With the normalization $\chi(\infty) = W_\infty$ we obtain for the successive determination of the χ_n

$$\left.\begin{aligned} \chi_0(y) &= W(y), \\ \chi_{n+1}(y) &= 2W(y)\int_\infty^y W^{-2}(y_1)\,dy_1 \int_\infty^{y_1} W(y_2)\,\chi_n'(y_2)\,dy_2. \end{aligned}\right\} \tag{2.6}$$

The analogues of (2.6) for the θ_n have ∞ replaced by $-\infty$ in the limits of integration. Explicit formulas for the first three χ_n, obtained from (2.6), are

$$\chi_0(y) = W(y), \tag{2.7$_0$}$$

$$\chi_1(y) = W(y)\int_\infty^y [1-(W_\infty/W)^2]\,dy, \tag{2.7$_1$}$$

$$\chi_2(y) = W(y)\int_\infty^y \left\{\int_\infty^{y_1} [1-(W_\infty/W)^2]\,dy_2 + W^{-2}\int_\infty^{y_1}[W^2 - W_\infty^2]\,dy_2\right\}dy_1. \tag{2.7$_2$}$$

Again, for the θ_n, we have $-\infty$ instead of ∞, $W_{-\infty}$ instead of W_∞.

We now consider the eigenvalue relation for a general unbounded flow. Since our normalization does not force $\chi(0) = \theta(0)$, we can say only that if the eigenfunction is taken as $\exp(-\alpha y)\,\chi(y)$ for $y > 0$ then its continuation to $y < 0$ must be a multiple of $\exp(\alpha y)\,\theta(y)$, i.e. we must have

$$\chi(0) = K\theta(0) \quad \text{and} \quad \chi'(0) - \alpha\chi(0) = K[\theta'(0) + \alpha\theta(0)],$$

where K may be a function of α and c. Eliminating K we obtain the condition

$$\theta(0)\,\chi'(0) - \chi(0)\,\theta'(0) - 2\alpha\theta(0)\,\chi(0) = 0, \tag{2.8}$$

an equation which is of course not an identity in α for fixed c, but the eigenvalue relation between c and α.

If the χ_n from (2.7$_n$) and the corresponding θ_n are now used in (2.8) we get for the eigenvalue relation, retaining terms up to order α^3,

$$\left.\begin{aligned} &-\alpha[W_\infty^2 + W_{-\infty}^2] - \alpha^2\int_{-\infty}^\infty (W^2 - W_\infty^2)(W^2 - W_{-\infty}^2)\,W^{-2}\,dy \\ &+\alpha^3\int_{-\infty}^\infty dy\int_{-\infty}^y (W^2(y) - W_\infty^2)(W^2(y_1) - W_{-\infty}^2)(W^{-2}(y) + W^{-2}(y_1))\,dy_1 + \ldots = 0. \end{aligned}\right\} \tag{2.9}$$

In the jet case this can be simplified to give

$$\left.\begin{aligned} &2c^2 + \alpha\left\{\int_{-\infty}^\infty (w^2 - 2cw)\,dy - c^2\int_{-\infty}^\infty (W^2 - c^2)\,W^{-2}\,dy\right\} \\ &-\alpha^2\left\{\int_{-\infty}^\infty (w^2 - 2cw)\,dy\right\}\left\{\int_{-\infty}^\infty (W^2 - c^2)\,W^{-2}\,dy\right\} + \ldots = 0. \end{aligned}\right\} \tag{2.9J}$$

Note that these formulas, although obtained by matching at $y = 0$, are independent of the choice of origin. (2.9) and (2.9J) come from (2.8), which expresses the vanishing at $y = 0$ of the Wronskian of the two solutions $\theta\exp(\alpha y)$ and $\chi\exp(-\alpha y)$ of the Rayleigh equation (2.1). Since the Wronskian of such an equation is constant, and the construction of θ and χ was independent of the origin, (2.9) must also be independent of the origin.

Examples of the use of (2.9) will be given in §§5 and 7. We note at this point, however, that the first approximation to (2.9) is $W_\infty^2 + W_{-\infty}^2 = 0$, i.e. taking the root corresponding to $c_i > 0$,

$$c = \tfrac{1}{2}\{w(\infty)+w(-\infty)\}+\tfrac{1}{2}i\{w(\infty)-w(-\infty)\}.$$

Thus, in the shear-layer case, this first approximation gives $c = i$, as we have already anticipated, and such flows are always unstable to long waves. In the jet case, $w(\infty) = w(-\infty) = 0$, and we must go to the next approximation. We have $c = 0$ to the first approximation, and then $c^2 = -\tfrac{1}{2}\alpha \int_{-\infty}^{\infty} w^2 dy$ to the second approximation, if we use the first approximation for c in the integral. Thus in the jet case we have

$$c \sim i\alpha^{\frac{1}{2}}\left\{\frac{1}{2}\int_{-\infty}^{\infty} w^2 dy\right\}^{\frac{1}{2}} \quad (\text{as } \alpha \to 0). \tag{2.10}$$

Again, the flow is unstable to long waves, although $c_i \to 0$ as $\alpha \to 0$. Prof. George Backus has pointed out to us that (2.10) can be obtained, except for a numerical factor, by a simple physical argument which gives it an interesting interpretation. The idea is that a long wave instability of the jet is expected to be essentially a sinusoidal displacement (indeed the perturbation stream function which we have just computed *is* approximately $w(y)$ for small α, which, as we observed above, corresponds to a lateral displacement), and instability arises because of the centifugal force associated with the curved path. If the streamline displacement is $\eta(t)\cos\alpha x$ the curvature is $-\alpha^2\eta\cos\alpha x$, and the centrifugal force associated with a particular streamline is $\alpha^2\eta\cos\alpha x\, w^2(y)$. Integrating over the jet we obtain $\alpha^2\eta\cos\alpha x \int_{-\infty}^{\infty} w^2 dy$. This force is resisted by the inertia of the fluid which we may readily estimate by assuming that the perturbed motion extends essentially a distance of a wavelength into the exterior of the jet. Thus the order of magnitude of the force is $\alpha^2\eta \int_{-\infty}^{\infty} w^2 dy.\ \alpha^{-1}$ per wavelength, and this accelerates a mass of order α^{-2} at the rate $\eta''(t)$. Taking η proportional to $\exp(\alpha c_i t)$ this gives

$$\alpha^2\eta \int_{-\infty}^{\infty} w^2 dy \frac{1}{\alpha} = O\left(\frac{1}{\alpha^2}\alpha^2 c_i^2 \eta\right)$$

or $c_i^2 = O\left(\alpha \int_{-\infty}^{\infty} w^2 dy\right)$. This argument can in fact be refined to give the exact coefficient $\tfrac{1}{2}$ in accordance with equation (2.10).

3. Convergence of the series for χ

We sketch here a proof of the convergence of $\Sigma\chi_n\alpha^n$. The proof for θ is essentially the same.

For convenience we use the new independent variable $\eta \equiv e^{-y}$ $(0 < \eta < 1)$, with subscript η to denote differentiation with respect to η. Equation (2.5) then gives

$$(\chi_{n+1}/W)_\eta = -2\eta^{-1}W^{-2}\int_0^\eta W(\chi_n)_{\eta_1}d\eta_1$$
$$= -2\eta^{-1}\left\{\int_0^\eta (\chi_n/W)_{\eta_1}d\eta_1 - \frac{1}{2}\int_0^\eta (W^2)_{\eta_1}d\eta_1\int_0^{\eta_1}(\chi_n/W)_{\eta_2}d\eta_2\right\} \quad (n \geqslant 1) \tag{3.1}$$

We now suppose that w is such that $|(W^2)_\eta| < A\eta^{a-1}$ for some positive constants $a-1$ and A. This is satisfied for most velocity profiles of interest. We also assume that the imaginary part c_i of c is not zero, and note that since w is real $|W^{-2}| < c_i^{-2}$. These inequalities enable us to prove from (3.1) by mathematical induction (taking $\chi_0 = W$) that

$$|(\chi_n/W)_\eta| \leqslant (A/ac_i^2)(2/a)^{n-1}\eta^{a-1}\sum_{r=0}^{n-1}\frac{1}{r!}(A\eta^a/2ac_i^2)^r$$
$$< (A/ac_i^2)(2/a)^{n-1}\exp(A/2ac_i^2) \quad (n \geqslant 1)$$

for $0 \leqslant \eta \leqslant 1$. From this it follows that

$$|\chi_n/W| \leqslant \int_0^\eta |(\chi_n/W)_{\eta_1}|d\eta_1 < (A/ac_i^2)(2/a)^{n-1}\exp(A/2ac_i^2) \quad (n \geqslant 1).$$

Thus a sufficient condition for convergence of $\Sigma\chi_n\alpha^n$ is $\alpha < \frac{1}{2}a$.

4. Integral equation approach

The above proof of the convergence of $\Sigma\chi_n\alpha^n$ unfortunately does not answer all questions connected with convergence; in particular it is not at once clear that the series (2.9) for the function of α and c whose vanishing gives the eigenvalue relation, is convergent in the range in which we wish to use it. We have proved the convergence of $\Sigma\chi_n\alpha^n$ for a fixed c with non-zero imaginary part. We then obtain from this a series, the left-hand side of (2.9) or the expansion of $[\theta\chi' - \chi\theta' - 2\alpha\theta\chi]_{y=0}$, say, which is again convergent in α for fixed c with imaginary part $\neq 0$. It is clear from the proof that this convergence is even uniform for c in a compact set excluding $c_i = 0$. Now we wish to set this series equal to zero to determine c as a function of α for small α. There will clearly be no difficulty here if c_i for the eigenvalue does not approach zero as $\alpha \to 0$, and this in fact occurs in the shear-layer case. But in the jet case we have (at least tentatively) equation (2.10) which indicates that $c_i \to 0$ as $\alpha \to 0$, and so casts doubt on the validity of the expansion. It is in fact possible to show that (2.9) does converge rapidly enough that our formulas are correct even when $c_i \to 0$, provided that $c_i \to 0$ slowly enough as $\alpha \to 0$, and the behaviour given by (2.10) does actually satisfy this condition of 'slowness'. Rather than discuss this, however, we shall obtain a similar result in the course of the presentation of the previously mentioned integral equation attack. For simplicity we restrict attention to the jet case, and look for the eigenvalue with the behaviour (2.10) as $\alpha \to 0$. It is possible, however, to use a modification of this approach in the shear-layer case also. Beside the fact that the difficulty with $c_i \to 0$ is more readily handled with the integral equation method, this method also has the advantage of giving explicit formulas for all terms of an expansion of the eigenvalue relation.

If we set $F \equiv \phi/W$, the Rayleigh equation (2.1) can be written as

$$\frac{d}{dy}\left[W^2\frac{dF}{dy}\right] - \alpha^2 c^4 W^{-2} F = \alpha^2 W^{-2}(W^4 - c^4)\,F.$$

Now it is possible to find explicitly the Green's function for the differential operator on the left-hand side of this equation, with the boundary conditions $F(\pm\infty) = 0$. This is

$$G(y, y_1) = -(2\alpha c^2)^{-1} \exp\left[-\alpha c^2 \operatorname{sgn}(y - y_1) \int_{y_1}^{y} W^{-2}\,dy_2\right].$$

With the aid of this Green's function we now invert the operator to obtain the following integral equation for F

$$F(y) = -(\alpha/2c^2) \int_{-\infty}^{\infty} K(y, y_1)\,F(y_1)\,dy_1, \tag{4.1}$$

with $$K(y, y_1) \equiv W^{-2}(y_1)\,[W^4(y_1) - c^4] \exp\left[-\alpha c^2 \operatorname{sgn}(y - y_1) \int_{y_1}^{y} W^{-2}\,dy_2\right]. \tag{4.2}$$

We now write down the Fredholm determinant for this integral equation

$$D \equiv 1 + (\alpha/2c^2) \int_{-\infty}^{\infty} K(y, y)\,dy + \frac{1}{2!}(\alpha/2c^2)^2 \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} \begin{vmatrix} K(y, y) & K(y, y_1) \\ K(y_1, y) & K(y_1, y_1) \end{vmatrix} dy_1 + \dots. \tag{4.3}$$

The vanishing of D gives the eigenvalue relation. Because of the infinite interval, we cannot simply appeal to the ordinary theorem to show that this series for D converges. However, it is possible to prove this directly in the present case, and we now sketch this proof. It proceeds on the assumptions that

$$|w| < A \exp(-a|y|)$$

for some positive constants A and a, and that c is 'almost pure imaginary' in the sense that $|c|/|c_i| \leqslant N < \infty$; thus we shall prove that the series for D converges if c is almost pure imaginary and then observe that the locus $D = 0$ does lie in this domain of convergence, being given in fact by (2.10) to first approximation, for small α.

The first step is to find explicit expressions for the determinants which occur in the series (4.3). For $1 \leqslant i, j \leqslant n$ we have

$$\det|K(y_i, y_j)| = W^{-2}(y_1)\,(W^4(y_1) - c^4) \dots W^{-2}(y_n)\,(W^4(y_n) - c^4) \det|k(y_i, y_j)|,$$

where $$k(y, y_1) \equiv \exp\left[-\alpha c^2 \operatorname{sgn}(y - y_1) \int_{y_1}^{y} W^{-2}\,dy_2\right],$$

for one factor $W^{-2}(W^4 - c^4)$ can be removed from each column of the original determinant. By interchange of a pair of rows and the corresponding columns it is easily seen that $k(y_1, \dots y_n) \equiv \det|k(y_i, y_j)|$ is symmetric in all n variables, since $k(y, y_1)$ is symmetric. Evidently $\det|K(y_i, y_j)|$ is also symmetric, so it is enough to consider its values for $y_1 \leqslant y_2 \leqslant \dots \leqslant y_n$; the integral over all the space of $y_1, \dots, y_n$ will be just $n!$ times the integral over this domain. But if $y_1 \leqslant y$,

$$k(y, y_1) = \exp\left(-\alpha c^2 \int_0^{y} W^{-2}\,dy_2\right) \Big/ \exp\left(-\alpha c^2 \int_0^{y_1} W^{-2}\,dy_2\right).$$

Using this in $k(y_1, \ldots, y_n)$ for $y_1 \leqslant y_2 \leqslant \ldots \leqslant y_n$, we are able to evaluate the determinant explicitly with the aid of elementary transformations. The result is

$$k(y_1, \ldots, y_n) = \left[1 - \exp\left(-2\alpha c^2 \int_{y_1}^{y_2} W^{-2} dy\right)\right]\left[1 - \exp\left(-2\alpha c^2 \int_{y_2}^{y_3} W^{-2} dy\right)\right] \ldots \left[1 - \exp\left(-2\alpha c^2 \int_{y_{n-1}}^{y_n} W^{-2} dy\right)\right] \quad (4.4)$$

(for $y_1 \leqslant y_2 \leqslant \ldots \leqslant y_n$).

Notice that (4.4) gives us an explicit formula for the nth term in the expansion of D.

We now apply (4.4) to estimate $k(y_1, \ldots, y_n)$. Writing $c = c_r + ic_i$, we have $|c^2 W^{-2}| \leqslant |c|^2 [(w - c_r)^2 + c_i^2]^{-1} < |c|^2/c_i^2$. Thus, by a very rough estimate,

$$\begin{aligned} |k(y_1, \ldots, y_n)| &\leqslant (1 + \exp[2\alpha|c|^2 c_i^{-2}(y_2 - y_1)]) \ldots (1 + \exp[2\alpha|c|^2 c_i^{-2}(y_n - y_{n-1})]) \\ &\leqslant 2^{n-1} \exp[2\alpha|c|^2 c_i^{-2}(y_n - y_1)] \end{aligned}$$

Dropping the condition $y_1 \leqslant y_2 \leqslant \ldots \leqslant y_n$, we see from the symmetry that, in any case,

$$|k(y_1, \ldots y_n)| \leqslant 2^{n-1} \exp\{2\alpha|c|^2 c_i^{-2}[|y_1| + |y_2| + \ldots + |y_n|]\},$$

and so

$$\begin{aligned} |\det| K(y_i, y_j)|| \leqslant 2^{n-1}|W^{-2}(y_1)(W^4(y_1) - c^4)| \exp(2\alpha|c|^2 c_i^{-2}|y_1|) \ldots \\ \ldots |W^{-2}(y_n)(W^4(y_n) - c^4)| \exp(2\alpha|c|^2 c_i^{-2}|y_n|). \end{aligned}$$

From this it is clear that (4.3) converges if

$$\int_{-\infty}^{\infty} |W^{-2}(y)(W^4(y) - c^4)| \exp(2\alpha|c|^2 c_i^{-2}|y|)\, dy < \infty$$

and we can show that this is the case if c is almost pure imaginary,

$$|w| < A \exp(-a|y|),$$

and α is small enough, as follows:

$$|W^{-2}(W^4 - c^4)| = |w^2 - 2wc| \, . \, |1 + c^2 W^{-2}| \leqslant |w| \, . \, |w - 2c| \, . \, [1 + |c|^2 c_i^{-2}] \leqslant B e^{-a|y|}$$

under our assumptions. From this, convergence of the integral, and so of (4.3), follows for small enough α.

The first two terms of (4.3) give us an approximation to the eigenvalue relation

$$1 + \frac{\alpha}{2c^2} \int_{-\infty}^{\infty} W^{-2}(W^4 - c^4)\, dy = 0.$$

Under our assumptions of almost pure imaginary c and $|w| < A \exp(-a|y|)$ it is possible by standard techniques to estimate this integral a little more closely than we have done above, and so show that

$$\int_{-\infty}^{\infty} W^{-2}(W^4 - c^4)\, dy = \int_{-\infty}^{\infty} (w^2 - 2cw)\, dy + O(|c|^2 \log |c|). \quad (4.5)$$

We thus obtain (2.10) more rigorously as the first approximation to $c(\alpha)$, and so verify that at least for one eigenfunction we do have c almost pure imaginary for small enough α.

Including the next term in (4.3) we have

$$1+\frac{\alpha}{2c^2}\int_{-\infty}^{\infty} W^{-2}(W^4-c^4)\,dy+\left(\frac{\alpha}{2c^2}\right)^2\int_{-\infty}^{\infty} W^{-2}(W^4-c^4)\,dy$$
$$\times\int_{-\infty}^{y} W^{-2}(W^4-c^4)\left[1-\exp\left(-2\alpha c^2\int_{y_1}^{y} W^{-2}\,dy_2\right)\right]dy_1 = 0.$$

With $c^2 = O(\alpha)$, (4.4) shows that $k(y_1, \ldots, y_n) = O(\alpha^{2(n-1)})$ and so, if we include those terms in (4.3) which involve n-tuple or lesser integrals we shall have our eigenvalue relation with error of order α^{2n}. In particular, then, up to but not including order α^4 we have

$$1+\frac{\alpha}{2c^2}\int_{-\infty}^{\infty} W^{-2}(W^4-c^4)\,dy+\left(\frac{\alpha}{2c^2}\right)^2\int_{-\infty}^{\infty} W^{-2}(W^4-c^4)\,dy$$
$$\times\int_{-\infty}^{y} dy_1\, W^{-2}(W^4-c^4)\,(2\alpha c^2)\int_{y_1}^{y} W^{-2}\,dy_2 = 0. \quad (4.6)$$

5. Examples

We list first exact solutions for various broken-line velocity profiles which have been given by Kelvin, Helmholtz and Rayleigh (cf. Rayleigh 1945) or are obtainable by the same method, namely, by solving (1.1) separately in intervals of continuity of w' and matching pressure and normal velocity at the discontinuity surfaces. Approximate formulas for small α are also given. Sketches of these profiles are shown in figure 1.

1. 'Trapezium' profile:

$$\left.\begin{aligned} w(y) &= 0, && |y| > 1,\\ &= 1-(|y|-a)/(1-a), && 1 > |y| > a,\\ &= 1, && a > |y|. \end{aligned}\right\} \quad (5.1)$$

Eigenvalue relation (Rayleigh 1945, p. 397):

$$4(1-a)^2\alpha^2c^2-2(1-a)\,\alpha c\{2(1-a)\,\alpha\mp\exp(-2a\alpha)\,[1-\exp\{-2(1-a)\,\alpha\}]\}$$
$$+\{-1+2(1-a)\,\alpha+\exp[-2(1-a)\,\alpha]\}$$
$$\mp\exp(-2a\alpha)\,\{1-[1+2(1-a)\alpha]\exp[-2(1-a)\,\alpha]\} = 0, \quad (5.2)$$

where the upper sign is for antisymmetric (sinuous) and the lower for symmetric (varicose) disturbances.

For small α

Antisymmetric:
$$c = i\alpha^{\frac{1}{2}}[a+\tfrac{1}{3}(1-a)]^{\frac{1}{2}}+\ldots. \quad (5.3)$$

Symmetric:
$$c = 1+i(a\alpha)^{\frac{1}{2}}+\ldots. \quad (5.4)$$

Special cases of particular interest are:

1*a*. $a = 1$ ('Rectangle' profile; cf. Rayleigh 1945, pp. 380, 381)

Antisymmetric:
$$c = [1+i(\coth\alpha)^{\frac{1}{2}}]/[1+\coth\alpha] = i\alpha^{\frac{1}{2}}+\ldots. \quad (5.5)$$

Symmetric:
$$c = [1+i(\tanh\alpha)^{\frac{1}{2}}]/[1+\tanh\alpha] = 1+i\alpha^{\frac{1}{2}}+\ldots. \quad (5.6)$$

1*b*. $a = 0$ ('Triangle' profile; cf. Rayleigh 1945, p. 395)
Antisymmetric:

$$2\alpha^2c^2+\alpha(1-2\alpha-e^{-2\alpha})\,c+\{\alpha(1+e^{-2\alpha})-1+e^{-2\alpha}\} = 0. \tag{5.7}$$

or

$$c = \tfrac{1}{3}i\alpha^{\frac{1}{2}}+\ldots.$$

Symmetric:

$$c = 1/2\alpha(1-e^{-2\alpha}) = 1-\alpha+\ldots. \tag{5.8}$$

2. 'Double-jet' profile:

$$\left.\begin{aligned} w(y) &= 0, \quad |y| > 1, \\ &= U, \quad -1 < y < 0, \\ &= 1, \quad 0 < y < 1. \end{aligned}\right\} \tag{5.9}$$

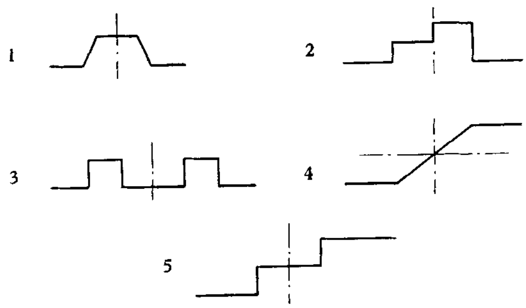

FIGURE 1. Broken line velocity profiles.

Eigenvalue relation ($T \equiv \tanh\alpha$):

$$(1-c)^2\,[c^2+(1-c)^2\,T]\,[c^2T+(U-c)^2]$$
$$+(U-c)^2[c^2T+(1-c)^2]\,[c^2+(U-c)^2\,T] = 0. \tag{5.10}$$

For small α there are three (conjugate pairs of) roots:

$$c = i[\tfrac{1}{2}(1+U^2)\,\alpha]^{\frac{1}{2}}+\ldots, \tag{5.11}$$
$$c = 1+i(\tfrac{1}{2}\alpha)^{\frac{1}{2}}+\ldots, \tag{5.12}$$
$$c = U[1+i(\tfrac{1}{2}\alpha)^{\frac{1}{2}}+\ldots]. \tag{5.13}$$

3. 'Separated double-jet' profile:

$$\left.\begin{aligned} w &= 0, \quad |y| > 2 \quad \text{and} \quad |y| < 1, \\ &= 1, \quad -2 < y < -1 \quad \text{and} \quad 1 < y < 2. \end{aligned}\right\} \tag{5.14}$$

Eigenvalue relations:
Antisymmetric:

$$\left(\frac{c}{1-c}\right)^2\left[1+\left(\frac{c}{1-c}\right)^2\tanh\alpha\right]+1+\left(\frac{c}{1-c}\right)^2\coth\alpha = 0. \tag{5.15}$$

Symmetric:

$$\left(\frac{c}{1-c}\right)^2\left[\left(\frac{c}{1-c}\right)^2+\coth\alpha\right]+\left(\frac{c}{1-c}\right)^2+\tanh\alpha = 0. \tag{5.16}$$

For small α there are four (conjugate pairs of) roots

Antisymmetric:

$$c = i\alpha^{\frac{1}{2}} + \dots, \tag{5.17}$$

$$c = 1 + i\alpha + \dots. \tag{5.18}$$

Symmetric:

$$c = 1 + i\alpha^{\frac{1}{2}} + \dots, \tag{5.19}$$

$$c = i\alpha + \dots. \tag{5.20}$$

4. Shear-layer or 'half-jet' profile:

$$\left.\begin{aligned} w(y) &= y/|y|, \quad |y| > a, \\ &= y/a, \quad\;\; |y| < a. \end{aligned}\right\} \tag{5.21}$$

Eigenvalue relation (Rayleigh 1945, p. 393):

$$c^2 = (4a^2\alpha^2)^{-1}\left[(1-2a\alpha)^2 - e^{-4a\alpha}\right]. \tag{5.22}$$

Only two special cases are of interest, $a = 1$ and $a = 0$, the latter giving the Helmholtz flow with eigenvalue relation $c^2 + 1 = 0$. For small α:

$$c = i(1 - \tfrac{4}{3}a\alpha + \dots). \tag{5.23}$$

5. 'Double shear-layer' profile:

$$\left.\begin{aligned} w(y) &= y/|y|, \quad |y| > 1, \\ &= 0, \qquad\;\; |y| < 1. \end{aligned}\right\} \tag{5.24}$$

Eigenvalue relation:

$$[c^4 + (1-c^2)^2]\tanh\alpha + 2c^2(1+c^2) = 0. \tag{5.25}$$

For small α there are two (pairs of) roots:

$$c = i(1 + \tfrac{1}{4}\alpha + \dots), \tag{5.26}$$

$$c = i(\tfrac{1}{2}\alpha)^{\frac{1}{2}}(1 + \tfrac{1}{4}\alpha + \dots). \tag{5.27}$$

Known results for other profiles are mainly restricted to the neutral solutions associated with inflexion points, and so are not really suitable for comparison with our formulas. However, our results for small α can be extended toward Lin's perturbation (1.4) of these neutral solutions, so for completeness we give some of these.

6. Bickley jet

$$w(y) = \operatorname{sech}^2 y. \tag{5.28}$$

Antisymmetric (Savic 1941):

$$\phi_s = \operatorname{sech}^2 y, \quad w_s = \tfrac{2}{3}, \quad \alpha_s = 2. \tag{5.29}$$

Symmetric (Savic & Murphy 1943):

$$\phi_s = \operatorname{sech} y \tanh y, \quad w_s = \tfrac{2}{3}, \quad \alpha_s = 1. \tag{5.30}$$

7. Antisymmetric double jets (Curle 1956*a*, *b*)

$$w(y) = \operatorname{sech}^m y \tanh y, \quad (m \geqslant -\tfrac{1}{2}), \tag{5.31}$$

$$\phi_s = \operatorname{sech}^{m+1} y, \quad w_s = 0, \quad \alpha_s = 2m+1. \tag{5.32}$$

These are neutral solutions corresponding to the inflexion points at $y = 0$. Solutions corresponding to the other inflexion points are apparently not known. The special case $m = 0$, a shear-layer profile, was also given by Garcia (1956).

8. Erf half-jet $w(y) = \operatorname{erf} y$. Carrier (cf. Esch 1957) has given some numerical results, including growth rates, for this case.

9. Wake profiles. Hollingdale (1940) has given numerical solutions for some wake profiles, including $w = \exp \frac{1}{2}(1-y^2)$, as well as for some shear-layer profiles.

We illustrate the application of our formulas (2.9) and (2.9J) with two of these examples. The first is of the jet type, the simple rectangular jet 1a above. (2.9J) gives for the eigenvalue relation,

$$2c^2+\alpha\left\{2(1-2c)-c^2\frac{2(1-2c)}{(1-c)^2}\right\}-\alpha^2\{2(1-2c)\}\left\{\frac{2(1-2c)}{(1-c)^2}\right\}+\ldots = 0,$$

or

$$2c^2+(1-2c)^2(1-c)^{-2}(2\alpha-4\alpha^2)+\ldots = 0.$$

Solving this for c, to the order of approximation indicated, we find

$$c = \alpha-\alpha^2+\ldots+i\alpha^{\frac{1}{2}}(1-\alpha+\ldots)$$

for the sinuous disturbance, and

$$c = 1-\alpha+\alpha^2+\ldots+i\alpha^{\frac{1}{2}}(1-\alpha+\ldots)$$

for the varicose disturbance. These are readily seen to agree with the exact solutions (5.5) and (5.6).

Our second example is of the shear-layer type, the half-jet profile 4 above. We take $a = 1$ in equation (5.21). The eigenvalue relation (2.9) then gives

$$-2\alpha(1+c^2)-\alpha^2(-16/3)+\alpha^3(-16/3)+\ldots = 0,$$

or

$$c^2 = -1+(8/3)\alpha-(8/3)\alpha^2+\ldots,$$

in agreement with Rayleigh's result (5.22).

Our series are used for the determination of the instability characteristics of the smooth profiles $w = \tanh y$ and $w = \operatorname{sech}^2 y$ in §7, and the results compared with previous numerical calculations.

6. General discussion

(a) *Stability characteristics*

We begin with an account of some stability characteristics for general values of wave-number. This and the following account of the characteristics for small wave-numbers obtained by using the formulas of the preceding sections will lead to a clear overall picture of the stability characteristics of unbounded flows.

Howard (1961) has shown that the locus of unstable eigenvalues (that is, the set of values $c(\alpha)$ with $c_i > 0$) can be limited in a general way. He proved that, if $c_i > 0$, c lies in the semi-circle in the complex plane defined by

$$[c_r-\tfrac{1}{2}(a+b)]^2+c_i^2 \leqslant [\tfrac{1}{2}(a-b)]^2, \tag{6.1}$$

where a is the maximum and b the minimum value of $w(y)$ over the field of flow. This semi-circle theorem is valid for unbounded, semi-bounded and bounded flows. It implies Rayleigh's theorem that c lies within the range of w. Howard's equation

(3.1) (1961, p. 510) incidentally implies that for a homogeneous fluid c_r also lies within the range of w when $c_i = 0$. (However, for a stratified fluid under gravity, internal waves with c_r outside the range of w are possible; these are isolated modes, not being the limit of unstable modes as $c_i \to 0$.) That the semi-circle theorem gives the best general estimate can be seen from our example 1 *a* of § 5 (the rectangular jet profile), for which the locus of unstable eigenvalues is exactly the semi-circle

$$(c_r - \tfrac{1}{2})^2 + c_i^2 = \tfrac{1}{4} \quad (c_i > 0)$$

with the exception of the point $c = \frac{1}{2}(1+i)$, which is approached as $\alpha \to \infty$.

We shall next prove that for many unbounded flows there is an upper bound on the values of α which can have eigenfunctions with $c_i > 0$, and we shall give an estimate of this upper bound which seems to be quite good. For the proof we make hypotheses that restrict w, but not very severely; examples of profiles which satisfy them are sufficiently smooth monotonic shear-layer profiles with just one point of inflexion and jet profiles with just two points of inflexion, both occurring at the same value of w. The hypotheses are that (*a*) w is of class C^2; (*b*) there exists a real number c_s such that $w''(w-c_s) < 0$ everywhere; and (*c*) $|w''/(w-c_s)|$ is both integrable and square integrable on $(-\infty, \infty)$.

We start with the Rayleigh stability equation in the form (2.1) and invert the operator $(D^2 - \alpha^2)$ to get

$$\phi(y) = -\frac{1}{2\alpha}\int_{-\infty}^{\infty} \exp(-\alpha|y-y_1|)\, w'' W^{-1}\phi\, dy_1. \tag{6.2}$$

(This can be done, for example, by variation of parameters.) With $F \equiv \phi/W$, this may be written

$$WF = -\frac{1}{2\alpha}\int_{-\infty}^{\infty} \exp(-\alpha|y-y_1|)\, w'' F\, dy_1. \tag{6.3}$$

If we multiply this equation by $w''\bar{F}$ and integrate, we get

$$\int_{-\infty}^{\infty} w''(w-c)\,|F|^2 dy = -\frac{1}{2\alpha}\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} w''(y)\bar{F}(y)\exp(-\alpha|y-y_1|)\, w''(y_1)F(y_1)\, dy_1,$$

where a bar denotes a complex conjugate. The right-hand side is obviously real, and the imaginary part of the left-hand side is $-c_i\int_{-\infty}^{\infty} w''|F|^2 dy$, which must thus be zero. On assuming that $c_i > 0$, we deduce that the integral must vanish. Thus we can replace c in the previous equation by any number. We choose to replace it by c_s, so that

$$\int_{-\infty}^{\infty} w''(w-c_s)\,|F|^2 dy = -\frac{1}{2\alpha}\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} w''(y)\bar{F}(y)\exp(-\alpha|y-y_1|)\, w''(y_1)F(y_1)\, dy_1.$$

Let us set $f \equiv [w''(w-c_s)]^{\frac{1}{2}} F$ and $k \equiv [-w''/(w-c_s)]^{\frac{1}{2}}$. Then, since

$$w''(w-c_s) < 0,$$

k is a non-negative real function, and the last equation can be written

$$\int_{-\infty}^{\infty} |f|^2 dy = \frac{1}{2\alpha}\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} \bar{f}(y)\, k(y) \exp(-\alpha|y-y_1|) f(y_1)\, k(y_1)\, dy_1. \quad (6.4)$$

But this implies that

$$\int_{-\infty}^{\infty} |f|^2 dy \leqslant \frac{1}{2\alpha}\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} |f(y)|\, k(y)\, |f(y_1)|\, k(y_1)\, dy_1$$

$$= \frac{1}{2\alpha}\left(\int_{-\infty}^{\infty} |f(y)|\, k(y)\, dy\right)^2 \leqslant \frac{1}{2\alpha}\int_{-\infty}^{\infty} |f|^2 dy \int_{-\infty}^{\infty} k^2\, dy.$$

Since $\int_{-\infty}^{\infty} |f|^2\, dy$ is clearly not zero if $F \neq 0$, we get

$$\alpha \leqslant \frac{1}{2}\int_{-\infty}^{\infty} k^2 dy = \frac{1}{2}\int_{-\infty}^{\infty} |w''/(w-c_s)|\, dy, \quad (6.5)$$

giving an upper bound on the possible wave-numbers of an unstable disturbance. (This result can also be shown to hold for bounded flow, under the same hypothesis $w''(w-c_s) \leqslant 0$.)

A different and sometimes better upper bound is obtained as follows. Equation (6.4) implies that

$$\int_{-\infty}^{\infty} |f|^2 dy \leqslant \frac{1}{2\alpha}\int_{-\infty}^{\infty} |f|\, k\, dy \int_{-\infty}^{\infty} |f(y)_1|\, k(y_1) \exp(-\alpha|y-y_1|)\, dy_1,$$

$$\leqslant \frac{1}{2\alpha}\int_{-\infty}^{\infty} |f|\, k\, dy \left\{\int_{-\infty}^{\infty} k^2(y_1) \exp(-2\alpha|y-y_1|)\, dy_1 \int_{-\infty}^{\infty} |f|^2 dy_2\right\}^{\frac{1}{2}},$$

$$\leqslant \frac{1}{2\alpha}\int_{-\infty}^{\infty} |f|^2 dy_2 \left\{\int_{-\infty}^{\infty} k^2(y)\, dy \int_{-\infty}^{\infty} k^2(y_1) \exp(-2\alpha|y-y_1|)\, dy_1\right\}^{\frac{1}{2}},$$

on repeated use of Schwarz's inequality. But

$$k^2(y)\, k^2(y_1) \leqslant \tfrac{1}{2}[k^4(y) + k^4(y_1)].$$

Therefore

$$1 \leqslant \frac{1}{2\alpha}\left\{\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} \tfrac{1}{2}[k^4(y) + k^4(y_1)] \exp(-2\alpha|y-y_1|)\, dy_1\right\}^{\frac{1}{2}},$$

$$= \frac{1}{2\alpha}\left\{\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} k^4(y) \exp(-2\alpha|y-y_1|)\, dy_1\right\}^{\frac{1}{2}},$$

$$= \frac{1}{2\alpha}\left\{\alpha^{-1}\int_{-\infty}^{\infty} k^4(y)\, dy\right\}^{\frac{1}{2}}.$$

Thus, when $c_i > 0$

$$\alpha^3 \leqslant \frac{1}{4}\int_{-\infty}^{\infty} k^4 dy = \frac{1}{4}\int_{-\infty}^{\infty} [w''/(w-c_s)]^2\, dy. \quad (6.6)$$

Then, at least in the cases covered by our hypotheses (*a*), (*b*), (*c*), sufficiently short waves are always stable. We naturally expect that the curve $c = c(\alpha)$ reaches the real axis at the values of c and α which correspond to the well-known neutral solution associated with the inflexion point of the velocity profile. It is thus of interest to compare the above estimates of the maximum unstable wave-number with some known results of neutral solutions. The half-jet of our example

7 ($w = \tanh y$) has a neutral solution at $\alpha = \alpha_s = 1$. The estimate (6.5) for this case gives $\alpha < 2$, while (6.6) gives $\alpha < (4/3)^{\frac{1}{3}} \doteqdot 1\cdot10$. The Bickley jet of example 6 ($w = \mathrm{sech}^2 y$) has two neutral solutions, with $\alpha_s = 1$ and 2 for the odd and even eigenfunctions, respectively. Estimate (6.5) gives $\alpha < 6$ for this case, and (6.6) gives $\alpha < 2(\frac{3}{2})^{\frac{1}{3}} \doteqdot 2\cdot29$. In a numerical comparison with Carrier's (cf. Esch 1957) computed results for the half-jet with $w = \mathrm{erf}\, y$, estimate (6.6) gives a result essentially indistinguishable from the computed value of $\alpha_s \doteqdot 1\cdot14$. These examples suggest that (6.6) gives a rather good estimate of the minimum wavelength of instability for ordinary unbounded flows. An estimate of α_s from *below* can be obtained by the use of a trial function in the variational statement of the eigenvalue problem with $c = c_s, \alpha = \alpha_s$. For the sinuous mode of an unbounded jet flow a convenient trial function is $\phi = w$, which gives the estimate

$$\alpha_s^2 \geqslant c_s^2 \left\{\int_{-\infty}^{\infty} [-w''/(w-c_s)]\, dy\right\} \left\{\int_{-\infty}^{\infty} w^2\, dy\right\}^{-1}.$$

For the Bickley jet this estimate happens to be exact.

(*b*) *Characteristics for small wave-numbers*

After these remarks on the general behaviour of the graph of $c = c(\alpha)$, we return to a discussion of the results of §§1–5 on the details of the behaviour as $\alpha \to 0$. Here we are interested in the case of $c_i > 0$ for $\alpha > 0$, although it is allowed that $c_i \to 0$ as $\alpha \to 0$. First, we may draw some plausible conjectures from the examples of broken-line profiles given in §5. It can be seen that for all the jet-type profiles $c_i \to 0$ as $\alpha \to 0$, while for the shear-layer type profiles sometimes $c_i \to 1$ as $\alpha \to 0$. Again, for the simplest types of jet, for instance, the trapezium profile (example 1), there are two modes of instability for each sufficiently small value of α (the familiar sinuous and varicose disturbances of a symmetrical jet). For one of these modes (the sinuous) $c_r \to 0$ as $\alpha \to 0$, while for the other $c_r \to \max w(y)$, $(-\infty < y < \infty)$. On the other hand, the double-jet profile of example 2 has three modes of instability for small enough α, one of which has $c \to 0$, while the other two have c approaching the values U and 1, which, one might guess, correspond to points at which $w' = 0$ in a similar smooth profile. In the simple shear-layer case of example 4 there is just one unstable wave for small α, and this has $c \to i$ as $\alpha \to 0$. In the double shear-layer of example 5, however, there are two unstable waves, one with $c \to i$ and another with $c \to 0$, which is again apparently to be identified with a value of w at which $w' = 0$. We are thus led to anticipate that the set of values of c which are limits of unstable waves for $\alpha \to 0$ consists of those values of w at points where $w' = 0$ and in addition zero (though this might be thought the limiting value of w as $y \to \pm\infty$, because $w' \to 0$ as $y \to \pm\infty$) in the jet case, and i in the shear-layer case.

We now prove that in any case the set of limiting values of c as $\alpha \to 0$ cannot be larger than the set we have just conjectured. We start from the Rayleigh stability equation in the form (1.6), namely

$$(W^2F')' = \alpha^2 W^2 F.$$

If we let $\alpha \to 0$ in this, we see that for the limiting F we must have $W^2F' =$ constant, and that this constant must in fact be zero to avoid divergence at infinity.

Thus the limiting F must be constant also, in intervals in which $W^2 \neq 0$, but it may have jumps at points at which W vanishes.

For the proof let us suppose first that the limiting F has no jumps and approaches 1, say, everywhere (non-uniformly at infinity, of course). This *must* happen if the limiting value of c is not real. Now, by differentiating the above form of the Rayleigh equation, we get

$$(W^2F')'' - \alpha^2(W^2F') = \alpha^2(W^2)'F;$$

an inversion of the differential operator on the left gives

$$W^2F' = -\tfrac{1}{2}\alpha \int_{-\infty}^{\infty} \exp(-\alpha|y-y_1|)(W^2)'F\,dy_1.$$

Differentiating this and using the Rayleigh equation, we get

$$W^2F = \alpha^{-2}(W^2F')' = \frac{1}{2}\int_{-\infty}^{\infty} \exp(-\alpha|y-y_1|)\,\mathrm{sgn}\,(y-y_1)(W^2)'F\,dy_1.$$

Letting $\alpha \to 0$ in this, we have for the case $F \to 1$

$$W^2(y) = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{sgn}\,(y-y_1)(W^2)'\,dy_1 = \tfrac{1}{2}[W^2(y) - W^2_{-\infty} - W^2_{\infty} + W^2(y)].$$

Therefore the limiting c must be such that $W^2_{\infty} + W^2_{-\infty} = 0$. In the jet case this gives $c \to 0$; in the shear-layer case $c \to i$. Thus i is the only possible non-real limit of c as $\alpha \to 0$, and can occur in the shear-layer case only.

Next suppose that the limiting F has a jump at the point $y = y_0$. We have seen above that W must vanish there. We shall now show that w' must vanish there also. Differentiating the integral form (6.3) of the Rayleigh equation, we get

$$WF' + w'F = \frac{1}{2}\int_{-\infty}^{\infty} \exp(-\alpha|y-y_1|)\,\mathrm{sgn}\,(y-y_1)\,w''F\,dy_1$$

Now let $\alpha \to 0$, taking y to have a value for which $W \neq 0$ in the limit (y_0 is one value of y at which $W = 0$, but there may be others). We thus obtain for the limiting F

$$w'F = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{sgn}\,(y-y_1)\,w''F\,dy_1,$$

this relation holding piece-wise between points at which $W = 0$. The right-hand side is easily seen to be continuous over $(-\infty, \infty)$, but the left-hand side has a jump at $y = y_0$ unless $w'(y_0) = 0$, which must accordingly be the case.

(c) *The eigenfunctions*

The above proof showed that i was the only possible non-real limit of c as $\alpha \to 0$, and that it could occur for the shear-layer case only. To show that it does occur, recall that in § 3 we proved convergence of our power series in α for θ and χ if c_i is bounded away from zero. As remarked in the first paragraph of § 4, we then get from these series a convergent series for the left-hand side of equation (2.8). We may therefore use our series in the eigenvalue relation (2.8) if it yields an eigenvalue $c(\alpha)$ which we find *ex post facto* to be non-zero in the limit $\alpha \to 0$, and in the shear-layer case we did indeed find the eigenvalue with $c_i \to 1$ as $\alpha \to 0$. In this case the limiting F has no jumps, since $w - i$ does not vanish. This is obvious also from the series for χ and θ. Thus in the shear-layer case we see that $c = c(\alpha)$ does have one branch at any rate which starts at $c = i$.

In §4 we showed that in the jet case there is a branch of $c = c(\alpha)$ with $c \to 0$ as $\alpha \to 0$, the behaviour for small α being given by the asymptotic relation (2.10). The integral equation (4.1), together with some simple estimates on the kernel, shows that in this case also the limiting F has no jumps, and calculations can be made with either (2.9), (2.9J), or the formulas of §4.

Next we show by construction that there do exist eigenfunctions for which c approaches (as $\alpha \to 0$) the value of w at a point y_0 where $w' = 0$, the limiting form of the function F now having a jump at this point. Let us suppose that $w'(y_0) = 0$, $w_0 \equiv w(y_0)$, and for the present also that $w''(y_0) \neq 0$ and $w_0 \neq w(\infty)$ or $w(-\infty)$. The simplest approach in this case seems to be to use our expansions to obtain representations of the solutions satisfying the boundary conditions at $y = \pm\infty$, but to use these expansions only sufficiently close to $\pm\infty$ that $(w - w_0)$ does not vanish in the ranges of integration involved. Thus suppose that $y = a$ is a point to the left, and $y = b$ a point to the right, of all zeros of $(w - w_0)$ on the real line. We then construct the solution $\chi \exp(-\alpha y)$ using equations (2.7), but we shall use this representation only for $y > b$, where there is no difficulty with convergence of the integrals as $c \to w_0$. Similarly we construct $\theta e^{\alpha y}$ for $y < a$. We now study the behaviour of the solutions of the Rayleigh stability equation between a and b as $\alpha \to 0$ and $c \to w_0$. For this purpose it is more convenient to work directly with Heisenberg's (cf. Lin 1955) series in powers of α^2 for F, since the formulas are simpler if we do not split off the exponential factors; it is permissible to do this if we stay inside the fixed finite interval $[a, b]$. The expansions of §2 are thus used to handle the non-uniformities at infinity, but we make a direct study of the singularity which occurs if $c \to w_0$ as $\alpha \to 0$.

The first step is to construct the solution F_1 of the Rayleigh equation (1.6),

$$(W^2F')' = \alpha^2 W^2 F, \tag{6.7}$$

which satisfies the boundary conditions at $y = a$,

$$F_1(a) = 1, \quad F_1'(a) = 0. \tag{6.8}$$

For F_1 we readily find the Volterra integral equation

$$F_1(y) = 1 + \alpha^2 \int_a^y [M(y) - M(y_1)]\, W^2(y_1) F_1(y_1)\, dy_1, \tag{6.9}$$

where $M(y) \equiv \int_a^y W^{-2} dy_1$. (We are assuming, of course, that $c_i > 0$.) Solving equation (6.9) by iteration in the usual way, we get

$$\begin{aligned} F_1(y) = 1 + \alpha^2 \int_a^y [M(y) - M(y_1)]\, W^2 dy_1 + \dots \\ + \alpha^{2n} \left\{ \int_a^y [M(y) - M(y_1)]\, W^2 dy_1 \int_a^{y_1} [M(y_1) - M(y_2)]\, W^2 dy_2 \dots \right. \\ \left. \dots \int_a^{y_{n-1}} [M(y_{n-1}) - M(y_n)]\, W^2 dy_n \right\} + \dots. \end{aligned} \tag{6.10}$$

Next we construct the solution F_2 which satisfies

$$F_2(a) = 0, \quad F_2'(a) = 1, \tag{6.11}$$

in a similar way. The result is

$$F_2(y)/W_a^2 = M(y) + \alpha^2 \int_a^y [M(y) - M(y_1)]\, W^2 M(y_1)\, dy_1 + \dots$$
$$+ \alpha^{2n} \left\{ \int_a^y [M(y) - M(y_1)]\, dy_1 \dots \right.$$
$$\left. \dots \int_a^{y_{n-1}} [M(y_{n-1}) - M(y_n)]\, W^2 M(y_n)\, dy_n \right\} + \dots, \qquad (6.12)$$

where $W_a \equiv W(a)$. (We shall use subscripts a and b to denote evaluation at these points.) These series (6.10), (6.12) are those of Heisenberg (cf. Lin 1955, p. 34).

We now take the solution $F = \exp\{\alpha(y-a)\}\theta/W$, which satisfies the boundary conditions at $y = -\infty$, and continue it into (a, b) by representing it as a linear combination of F_1 and F_2. This gives

$$F = (\theta/W)_a F_1 + [\alpha(\theta/W) + (\theta/W)']_a F_2.$$

For an eigenfunction, this must be proportional to the solution which satisfies the boundary conditions at $y = +\infty$, namely $\exp\{-\alpha(y-b)\}\chi/W$. Therefore the eigenvalue relation is

$$\{(\theta/W)_a F_1 + [\alpha(\theta/W) + (\theta/W)']_a F_2\}_b \{-\alpha(\chi/W) + (\chi/W)'\}_b$$
$$= \{(\theta/W)_a F_1' + [\alpha(\theta/W) + (\theta/W)']_a F_2'\}_b (\chi/W)_b. \quad (6.13)$$

We wish to show that this equation can be satisfied with a suitable $c(\alpha)$ which approaches w_0 as $\alpha \to 0$. The behaviour of χ and θ for small α and $(c - w_0)$ is readily obtained from equations (2.9) and their analogues for θ. To study equation (6.13) we thus need to find the behaviour of F_1, F_2 and their derivatives at $y = b$. To do this, equations (6.10) and (6.12) show that the important thing to know is the behaviour of $M(y)$ as $c \to w_0$. We proceed to investigate this.

We define $\gamma \equiv c - w_0$, and suppose that $\gamma \to 0$ 'non-tangentially', i.e. inside an angle $\epsilon < \arg\gamma < \pi - \epsilon$ for some $\epsilon > 0$. We shall verify afterwards that the $c(\alpha)$ we obtain from the eigenvalue relation (6.13) has this property. Now $(w - w_0)$ may vanish at several points in $[a, b]$ in addition to the point y_0 where $w' = 0$. We shall suppose first that $w' \neq 0$ at each of these points, if any. In this case, as we shall verify in a moment, the essential behaviour of $M(y)$ for small γ is determined only by the neighbourhood of y_0; if there are two or more points at which w' and $(w - w_0)$ both vanish, additional complications arise which we shall indicate later. We first examine the behaviour of $M(y)$ in the neighbourhood of a point y_n at which $w = w_m = w_0$ and $w' \neq 0$. To do this, consider

$$M_n(y) = \int_{y_n-\delta}^{y} W^{-2}\, dy_1, \quad \text{for} \quad y_n - \delta \leqslant y \leqslant y_n + \delta,$$

say, with some $\delta > 0$ and small enough that $w' \neq 0$ throughout the interval $[y_n - \delta, y_n + \delta]$. Now, by the mean value theorem, we have, for some number η between y_n and y,

$$\frac{1}{w'(y)} = \frac{1}{w'(y_n)} + (w - w_m)\left[\frac{d}{dw}\left(\frac{1}{w'}\right)\right]_{y_n} + (w - w_m)^2 \left[\frac{d^2}{dw^2}\left(\frac{1}{w'}\right)\right]_{\eta}$$
$$= \frac{1}{w'(y_n)} - \frac{w - w_m}{w'^3(y_n)} + O(\gamma^2).$$

Therefore

$$M_n(y) = \int_{y_n-\delta}^{y} \frac{1}{w'(y_1)} \frac{w'(y_1)}{W^2}\, dy_1$$

$$= \frac{1}{w'(y_n)}\left[-\frac{1}{w-c}\right]_{y_n-\delta}^{y} - \frac{1}{w'^3(y_n)}\left[\log(w-c) - \frac{\gamma}{w-c}\right]_{y_n-\delta}^{y} + O(1),$$

since $|W^{-2}| = O(\gamma^{-2})$ if γ lies inside the angle $\epsilon < \arg\gamma < \pi - \epsilon$, as we have assumed. Therefore

$$M_n(y) = -\frac{1}{(w-c)\,w'(y_n)} - \frac{1}{w'^3(y_n)}\log|w-c| + O(1). \qquad (6.14)$$

Thus $M_n(y)$ becomes as large as $O(\gamma^{-1})$ in the neighbourhood of y_n, but returns to $O(1)$ at $y_n + \delta$.

We next consider the behaviour of $M(y)$ in the neighbourhood of y_m, a point at which $w' = 0$. Using subscripts m to denote evaluation at y_m, we suppose that $w''_m = -2k^2 < 0$, for definiteness, and suppose that $w''(y) < 0$ on $[y_m - \delta, y_m + \delta]$, where $\delta > 0$ is sufficiently small. Then

$$M_m(y) \equiv \int_{y_m-\delta}^{y} W^{-2}\, dy = \int_{y_m-\delta}^{y} (w - w_m - \gamma)^{-2}\, dy.$$

We define $v(y)$ by the equation $w - w_m = -k^2v^2$, with the choice of sign such that $(y - y_m)\,v(y) \geqslant 0$. It can then be readily shown that $v_m = 0$, $v'_m = 1$, and that $v' > 0$ on $[y_m - \delta, y_m + \delta]$. If we set $\sigma^2 = \gamma k^{-2}$, taking $\frac{1}{2}\epsilon < \arg\sigma < \frac{1}{2}\pi - \frac{1}{2}\epsilon$, we then find

$$k^4 M_m(y) = \int_{y_m-\delta}^{y} (\sigma^2 + v^2)^{-2}\, dy = \int_{y_m-\delta}^{y} \frac{1}{v'(y)} \frac{v'\, dy}{(v^2+\sigma^2)^2}.$$

Next we set

$$1/v' = a_0 + a_1 v + a_2 v^2 + a_3 v^3 + O(v^4),$$

and in a manner similar to that used in estimating $M_n(y)$ we find

$$\begin{aligned} k^4 M_m(y) = a_0\left\{\frac{1}{4i\sigma^3}\left[\log\left|\frac{v-i\sigma}{v+i\sigma}\right| + i\beta\right] + \frac{1}{2\sigma^2}\frac{v}{v^2+\sigma^2}\right\} \\ - \frac{\frac{1}{2}a_1}{v^2+\sigma^2} + a_2\left\{\frac{1}{4i\sigma}\left[\log\left|\frac{v-i\sigma}{v+i\sigma}\right| + i\beta\right] - \frac{1}{2}\frac{v}{v^2+\sigma^2}\right\} \\ + \tfrac{1}{2}a_3 \log(v^2+\sigma^2) + O(1). \end{aligned} \qquad (6.15)$$

Here $\beta \equiv \arg(v - i\sigma)/(v + i\sigma)$ is chosen so that it approaches zero as $v \to -\infty$ and approaches 2π as $v \to +\infty$ (note that $i\sigma$ lies in the second quadrant). The coefficients are

$$a_0 = 1, \quad a_1 = w'''_m/6k^2, \quad a_2 = 5w'''^2_m/96k^4 + w^{\text{iv}}_m/16k^2,$$
$$a_3 = w^{\text{v}}_m/60k^2 + w'''_m w^{\text{iv}}_m/24k^4 + w'''^3_m/54k^6$$

as is easily determined from $w - w_m = -k^2v^2$.

Equations (6.14) and (6.15) show that the function $M(y)$ for small γ can be described, rather loosely as follows. At the points like y_n, where $(w - w_m)$ vanishes but w' does not, it has large localized peaks, and at y_m it has a large jump, essentially of magnitude $\pi/2\sigma^3k^4$, which is of even larger order of magnitude than the peaks at points such as y_n.

In the eigenvalue relation (6.13) we need the values of F_1, F_2 and their derivatives at $y = b$. If we use just the leading terms of series (6.10) and (6.12) and their derivatives, we obtain for small γ (and small α)

$$\left.\begin{aligned} &F_1(b) \to 1, \quad F_1'(b) \sim \alpha^2 W_b^{-2}\int_a^b W^2 dy, \\ &F_2(b) \sim W_a^2\pi/2\sigma^3k^4, \quad F_2'(b) \sim W_a^2 W_b^{-2}. \end{aligned}\right\} \tag{6.16}$$

From equations (2.7) and the analogous equations for θ we see that for small α,

$$\left.\begin{aligned} &(\chi/W)_b = 1+O(\alpha); \quad (\chi/W)_b' - \alpha(\chi/W)_b = -\alpha W_\infty^2 W_b^{-2} + O(\alpha^2), \\ &(\theta/W)_a = 1+O(\alpha); \quad (\theta/W)_a' + \alpha(\theta/W)_a = \alpha W_{-\infty}^2 W_a^{-2} + O(\alpha^2). \end{aligned}\right\} \tag{6.17}$$

Putting relations (6.16) and (6.17) into the eigenvalue relation (6.13), and retaining only the largest terms, we get

$$\{1+\alpha W_{-\infty}^2\pi/2\sigma^3k^4\}\{-\alpha W_\infty^2 W_b^{-2}\} \sim \{\alpha W_{-\infty}^2 W_b^{-2}\}.$$

Therefore
$$\pi\alpha/2\sigma^3k^4 \sim -(W_\infty^2 + W_{-\infty}^2)/W_\infty^2 W_{-\infty}^2. \tag{6.18}$$

Thus if $\sigma \to 0$ as $\alpha \to 0$ in such a way that relation (6.18) holds, it appears that we shall be able to satisfy the eigenvalue relation (6.13). Recalling that $\sigma^2 = \gamma/k^2$, we obtain as the first term in $\gamma(\alpha)$,

$$\gamma = c - w_m \sim e^{\frac{2}{3}\pi i}\left[\frac{\pi\alpha}{2k}\frac{W_\infty^2 W_{-\infty}^2}{W_\infty^2 + W_{-\infty}^2}\right]^{\frac{2}{3}}. \tag{6.19}$$

With $\gamma = O(\alpha^{\frac{2}{3}})$ we can now return to equations (6.10) and (6.12) and check that the asymptotic relations (6.16) are really correct; this is not difficult to verify with the aid of expressions (6.14) for $M_n(y)$ and (6.15) for $M_m(y)$. We find

$$F_1(b) = 1+O(\alpha), \quad F_1'(b) = O(\alpha^2),$$
$$F_2(b) = \pi W_a^2/2k^4\sigma^3 + O(1), \quad F_2'(b) = W_a^2 W_b^{-2} + O(\alpha),$$

which show that relation (6.18) is in fact correct to order α. We have thus verified *ex post facto* our assumption that $\epsilon < \arg\gamma < \pi - \epsilon$ as $\alpha \to 0$. The logical basis of our proof is simply that we have produced a function and a relation $c = c(\alpha)$ and have verified that they are an eigenfunction and eigenvalue relation as $\alpha \to 0$. We used the formally derived expansions (6.10), (6.12) for the eigenfunction and the assumption about γ only as a guide to produce the function and relation.

In writing the first term (6.19) for γ we have, of course, taken W_∞ to denote $w(\infty) - w_m$, but, since relation (6.18) is correct to order α and $W_\infty = w(\infty) - w_m + \gamma$ differs from its limiting value by a term of order $\alpha^{\frac{2}{3}}$, we can in fact get the next term from it. In this way we get

$$\gamma = c - w_m = e^{\frac{2}{3}\pi i}\left[\frac{\pi\alpha}{2k}\frac{W_{\infty m}^2 W_{-\infty m}^2}{W_{\infty m}^2 + W_{-\infty m}^2}\right]^{\frac{2}{3}}$$
$$\times\left\{1 - \tfrac{4}{3}e^{\frac{2}{3}\pi i}\left[\frac{\pi\alpha}{2k}\right]^{\frac{2}{3}}\left[\frac{W_{\infty m}^2 W_{-\infty m}^2}{W_{\infty m}^2 + W_{-\infty m}^2}\right]^{\frac{5}{3}}(W_{\infty m}^{-3} + W_{-\infty m}^{-3}) + O(\alpha)\right\}, \tag{6.20}$$

where $W_{\infty m} \equiv w_\infty - w_m$, $W_{-\infty m} \equiv w_{-\infty} - w_m$.

We must now fill in three gaps in our proof. First, we have assumed that $w_m'' < 0$, so that the point y_m is a local *maximum*. Clearly nothing need be changed much if $w_m'' > 0$. In this case we should define k by $w_m'' \equiv 2k^2$, σ by $\sigma^2 \equiv -\gamma/k^2$

(with σ in the fourth quadrant), and v by $w-w_m = k^2v^2$ (with $(y-y_m)v > 0$ as before); the remaining steps are essentially the same as before, and we again find relation (6.20) except that the factors $\exp(\frac{2}{3}\pi i)$ are replaced by $\exp(\frac{1}{3}\pi i)$, thus changing the sign of the real part. This is in accord with the semi-circle theorem (6.1), since if w_m is the *minimum* of w, c_r must increase to keep inside the semi-circle as α increases from zero.

Secondly, we should examine the case $w''_m = 0$. Here the asymptotic behaviour of M_m differs, but can be investigated similarly if we suppose, say, that $w'_m = w''_m = \ldots = w_m^{(r-1)} = 0$, $w_m^{(r)} \neq 0$. It is perhaps sufficient to state that the results are similar to those we have just given, for the case $r = 2$, except that

$$c - w_m = O(\alpha^{r/(2r-1)})$$

instead of $\alpha^{\frac{2}{3}}$. This is consistent with the relation $c-w_m = O(\alpha^{\frac{1}{2}})$ for the varicose mode of the broken-line jet of example 1, which case presumably corresponds to $r \to \infty$.

Thirdly, we have assumed that y_m is the only point at which $w-w_m$ and w' both vanish. If there are two such points, it is easy to carry through the argument as above; allowance being made for both of the jumps of F_2. This leads to a mode with $c-w_m = O(\alpha^{\frac{2}{3}})$. However, one might expect the occurrence of another mode, since there would be two if w had slightly different values at the two points at which w' vanishes. The separated double jet of example 3 is instructive in this connexion. There we find four unstable modes for small α: a sinuous mode with $c \to 0$, as usual for jet-type flows; two varicose modes with $c \to 0$ and $c \to 1$, corresponding presumably to the minimum in the centre and the double maximum at $w = 1$; and finally another sinuous mode with $c \to 1$. Note, however, that the latter mode is qualitatively different from the other three in that it has $c-w_m = O(\alpha)$ rather than $O(\alpha^{\frac{1}{2}})$. Examination of the eigenfunctions shows that in the limit $\alpha \to 0$ F has equal 'jumps' at the two maxima $w = 1$ for the varicose mode with $c \to 1$ and 'jumps' of opposite sign but equal magnitude at these places for the sinuous mode with $c \to 1$. (With broken-line profiles like this double jet the 'jumps' in the limiting F are actually linear increases spread over the segment on which w takes its maximum value.) The situation with smooth profiles having a double maximum or minimum is similar. In addition to the mode with $c = w_m + O(\alpha^{\frac{2}{3}})$, for which the limiting F has two jumps as indicated above, there is another mode with $c = w_m + O(\alpha^{\frac{4}{3}})$. The limiting F for this mode is different from zero only between the two maxima, where it is constant.

We have thus shown that for fairly general smooth unbounded profiles there is one mode of instability corresponding to each zero y_m of w'. In the limit as $\alpha \to 0$, $c \to w_m$, except for the mode corresponding to the zero of w' at infinity in a shear layer, when $c \to i$. Then curves $c = c(\alpha)$ in the complex c-plane originate for $\alpha = 0$ on the real axis and proceed into the upper half-plane, always staying inside the semi-circle (6·1), whose diameter is the range of w. As α increases, each curve presumably returns to the real axis (and we have proved this is some cases), the return occurring at a value of α satisfying both inequalities (6.5), (6.6). This gives the number of neutral oscillations for $\alpha > 0$. If the zeros of w' are all simple, it follows by Rolle's theorem that each of the neutral oscillations is of the form found by Tollmien (1935) to exist, namely that with $c = w(y_s)$, $\alpha = \alpha_s \neq 0$, where y_s is a zero of w''.

This gives an outline of the stability characteristics of unbounded parallel flows.

7. Two numerical examples

In this final section we give numerical results obtained by applying our general formulas to two specific stability problems, the tanh and sech² profiles. Some numerical calculations for these or closely related profiles have been reported previously, so by comparison it is possible to get some idea of the range in which the power series will be useful for computation.

$w = \tanh y$ profile

In this case $w(y)$ is odd and c is pure imaginary for small α. Using the first three terms of our formulas we find, after evaluating the integrals, that

$$c_i = 1 - 1{\cdot}785\alpha + 1{\cdot}526\alpha^2 + \ldots.$$

Lin's perturbation (1.4) gives for α near 1

$$c_i = (2/\pi)(1-\alpha^2) + \ldots.$$

Lessen & Fox (1955) have given some calculations for the 'half-jet' profile (cf. Lin 1953), which with proper renormalization is quite similar to the $\tanh y$ profile, although it is not exactly antisymmetrical and c is accordingly not exactly pure imaginary. Carrier (cf. Esch 1957) has computed the growth rate for the error function profile, which again is quite similar to the $\tanh y$ profile. In figure 2 we show the growth rate αc_i as a function of α obtained from the calculations of Lessen & Fox and Carrier, by renormalizing in accordance with our conventions and so that in each case the neutral solution associated with the inflexion point occurs at wave-number $\alpha = 1$. On the same graph are shown the slope given by Lin's perturbation at $\alpha = 1$ and the growth rates given by the power series when respectively two and three terms are retained.

$w = \operatorname{sech}^2 y$ profile

For this case the three terms indicated explicitly in the eigenvalue relation (2.9) or (2.9J) reduce to

$$2c^2 + \alpha(\tfrac{4}{3} - 4c - c^2J) - \alpha^2(\tfrac{4}{3} - 4c)J + \ldots = 0,$$

where $J \equiv \int_{-\infty}^{\infty} (W^2 - c^2)\, W^{-2}\, dy$. For the sinuous mode, to be consistent with the terms dropped we need J only approximately and to the necessary order of approximation $J = 1 + \tfrac{1}{2}\log(24\alpha^{-1}) + \tfrac{1}{2}\pi i$. Solving for c to this order of approximation we find

$$c = \alpha + i\{\tfrac{2}{3}\alpha - \alpha^2 - \tfrac{1}{6}\alpha^2 \log(24\alpha^{-1}) - \tfrac{1}{6}\alpha^2\pi i\}^{\frac{1}{2}}.$$

Lin's perturbation for this case gives

Sinuous mode:

$$\partial c/\partial\alpha^2 = 0{\cdot}0423 - (0{\cdot}0278)\,i \quad (\text{at } \alpha = 2),$$

Varicose mode:

$$\partial c/\partial\alpha^2 = -0{\cdot}0264 - (0{\cdot}0835)\,i \quad (\text{at } \alpha = 1).$$

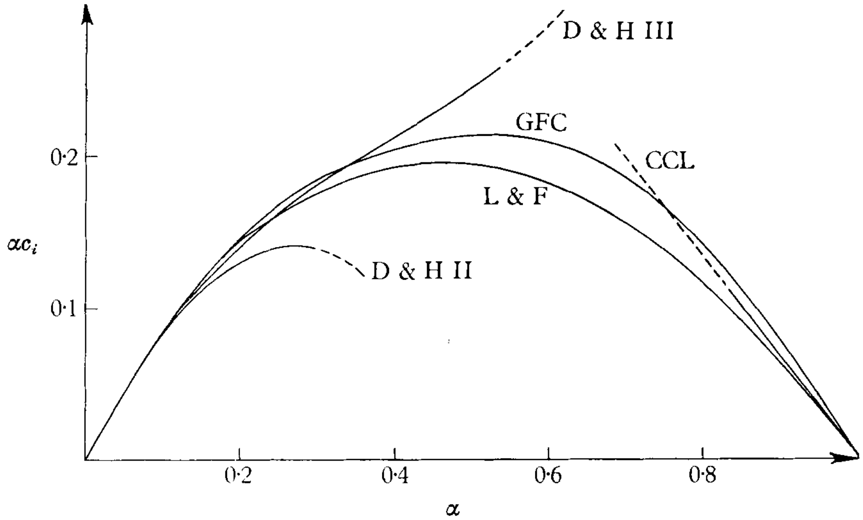

FIGURE 2. Shear-layer growth rates. The respective curves are: CCL—Lin's perturbation, tanh profile; GFC—Carrier's calculation for erf profile; L & F—Lessen & Fox's calculation for 'half-jet' profile; D & H, II, III—2- and 3-term approximations from power series, tanh profile.

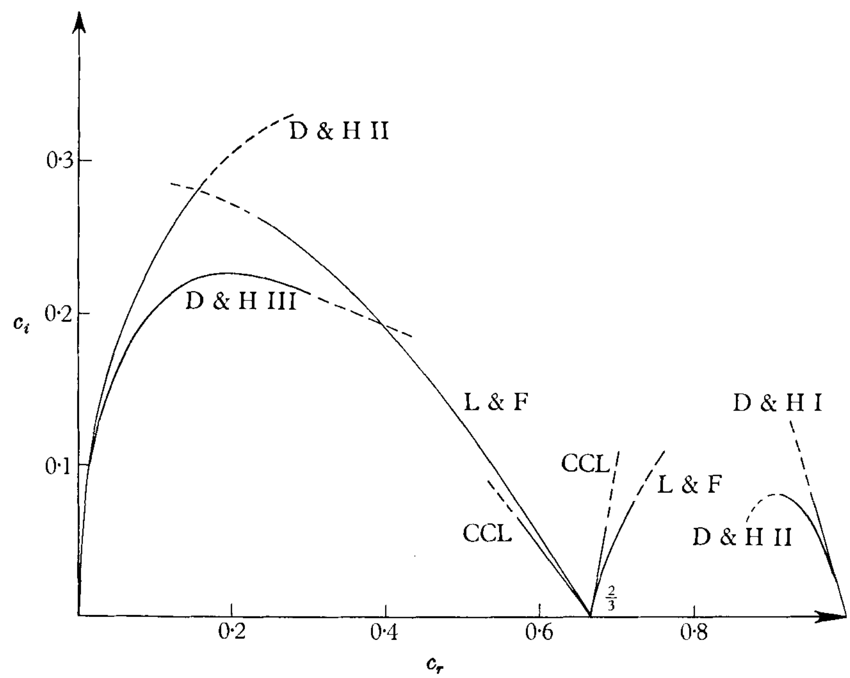

FIGURE 3. The complex wave speed for the sech² jet. CCL—Lin's perturbation; L & F—Lessen & Fox's calculation; D & H, I, II, III—1-, 2- and 3-term approximations from power series.

Lessen & Fox (1955) have also given some computations for both the sinuous and varicose modes in this case. In figure 3 we plot c_r versus c_i from their calculation and as obtained from our formula, with both the 2- and 3-term approximations. The slopes given by Lin's perturbation and the behaviour of the varicose mode near $\alpha = 0$, $c = 1$ given by the 1- and 2-term approximations of equations (6.19) and (6.20) are also indicated.

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